Astron 329/429 Fall 2015 midterm sol^{ns}

1. Radiation-dominated $\Rightarrow a(t) = (t/t_0)^{1/2}$

In general, $\rho_m = \rho_{m,0}/a^3$

$$\Rightarrow \boxed{\rho_m = \rho_{m,0}(t/t_0)^{-3/2}}.$$

2. $$\rho_{crit} = \frac{3H_0^2}{8\pi G}; \quad \rho_m(z) = \underbrace{\Omega_m \rho_{crit}}_{\rho_{m,0}}(1+z)^3$$

$$\rho_{rad,0} = \frac{\varepsilon_{rad}}{c^2} = \frac{\pi^2 K_B^4 T_{CMB}^4}{15\hbar^3 c^5}; \quad \rho_{rad}(z) = \rho_{rad,0}(1+z)^4$$

$$\rho_m(z) = \rho_{rad}(z) \Leftrightarrow \Omega_m\left(\frac{3H_0^2}{8\pi G}\right) = \frac{\pi^2 K_B^4 T_{CMB}^4}{15\hbar^3 c^5}(1+z)$$

$$\Leftrightarrow 1 + z_{eq} = \frac{15\hbar^3 c^5 \Omega_m}{\pi^2 K_B^4 T_{CMB}^4}\left(\frac{3H_0^2}{8\pi G}\right)$$

$$\Rightarrow z_{eq} \simeq 357.$$

3. a) Flat Friedmann: $$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda) = \frac{8\pi G}{3}\left(\frac{\rho_{m,0}}{a^3} + \underset{\text{const.}}{\rho_\Lambda}\right) \to \frac{8\pi G \rho_\Lambda}{3} = \text{const.} \equiv H_\infty$$ as $t \to \infty$.

$$\left[\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{crit,0}} = \frac{8\pi G \rho_\Lambda}{3H_0^2}, \Rightarrow \rho_\Lambda = \frac{3\Omega_\Lambda H_0^2}{8\pi G}\right]$$

b) $$\frac{da/dt}{a} = H_\infty \Rightarrow \ln a = H_\infty t + \text{const.}, \Rightarrow a(t) = A\exp(H_\infty t)$$
$$= a(t_1)\exp[H_\infty(t - t_1)]$$
for any t_1 large enough to be in Λ-dominated limit.

c) $\dot{a}(t) = a(t_1) H_\infty \exp[H_\infty(t - t_1)] \to \infty$ as $t \to \infty$ expansion accelerates.

4. a) $d_{com}(z) = -\int_{t_0}^{t(z)} \frac{c\,dt}{a(t)}$ $\qquad \boxed{\frac{1}{a}\frac{da}{dt} = H_0(\underbrace{\Omega_m}_{1} a^{-3})^{1/2} = H_0 a^{-3/2}}, \Rightarrow dt = \frac{a^{1/2}\,da}{H_0}$

$$= \int_{\frac{1}{1+z}}^{1} \frac{c\,da}{H_0 a^{1/2}} = \frac{c}{H_0}\left[\frac{a^{1/2}}{1/2}\right]_{\frac{1}{1+z}}^{1} = \frac{2c}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right]$$

$d_{com}(z_{LSS} = 1{,}000) \simeq 8{,}300$ Mpc

$d_{lum}(z_{LSS} = 1{,}000) = d_{com}(z_{LSS})(1+z_{LSS}) \simeq 8.3\times 10^6$ Mpc

$d_{diam}(z_{LSS} = 1{,}000) = \frac{d_{com}(z_{LSS})}{(1+z_{LSS})} \simeq 8.3$ Mpc

b) For $\Omega_\Lambda > 0$: $d_{com}(z) = \int_{\frac{1}{1+z}}^{1} \frac{c\,da}{a^2(\underbrace{\Omega_m a^{-3} + \underbrace{\Omega_\Lambda}_{1-\Omega_m}}_{(1)})^{1/2}}$ $\qquad \frac{1}{a}\frac{da}{dt} = H_0(\Omega_m a^{-3} + \Omega_\Lambda)^{1/2}$

$\Rightarrow dt = \frac{da}{a(\Omega_m a^{-3} + \Omega_\Lambda)^{1/2}}$

compare w/ matter-dominated: $\int_{\frac{1}{1+z}}^{1} \frac{c\,da}{a^2(\underbrace{1\,a^{-3} + 0}_{(2)})^{1/2}}$

(1) − (2): $\Omega_m a^{-3} + (1-\Omega_m) - a^{-3} = a^{-3}(\Omega_m - 1) + (1-\Omega_m) = \underbrace{(1-\Omega_m)}_{>0}\underbrace{(1-a^{-3})}_{<0} < 0$

$\Rightarrow d_{com}(z)$ is always larger when $\Omega_\Lambda > 0$ (assuming $\Omega_m + \Omega_\Lambda = 1$).

$\Rightarrow d_{diam}(z) = \frac{d_{com}(z)}{(1+z)}$ is larger at fixed z

$l = \theta\, d_{diam} \Rightarrow \theta = \frac{l}{d_{diam}}$, $\Rightarrow$ objects appear smaller when $\Omega_\Lambda > 0$.

5. When relativistic, $E_\nu(z) \simeq \frac{3}{2} K_B T_\nu(z_{ep}) \frac{(1+z)}{(1+z_{ep})}$

NR condition: $E_\nu(z_{nr}) = 2m_\nu c^2 = \frac{3}{2}K_B \underbrace{T_\nu(z_{ep})}_{\left(\frac{4}{11}\right)^{1/3} T_\gamma(z_{ep})} \frac{(1+z_{nr})}{(1+z_{ep})} = \frac{3}{2}\left(\frac{4}{11}\right)^{1/3} K_B T_\gamma(z=0)(1+z_{nr})$

$\Rightarrow z_{nr} = \frac{2m_\nu c^2}{K_B T_{CMB}(z=0)} \cdot \frac{2}{3}\left(\frac{11}{4}\right)^{1/3} - 1 \simeq 787$